

INFINITE ORDER DECOMPOSITIONS OF C*-ALGEBRAS

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Keywords: C*-algebra, Peirce decomposition, von Neumann algebra**Abstract**

In the given article infinite order decompositions of C*-algebras are investigated. It is proved that for the infinite order decomposition $\sum_{\xi, \eta}^{\oplus} p_{\xi} A p_{\eta}$ of a C*-algebra A with respect to an infinite orthogonal set $\{p_i\}$ of projections of A , if $p_{\xi} A p_{\xi}$ is a von Neumann algebra for any ξ then $\sum_{\xi, \eta}^{\oplus} p_{\xi} A p_{\eta}$ is a von Neumann algebra. Also, it is proved that, if a C*-algebra A with an infinite orthogonal set $\{p_{\xi}\}$ of projections in A such that $\sup_{\xi} p_{\xi} = 1$ is not a von Neumann algebra, projections of the set $\{p_{\xi}\}$ are pairwise equivalent then $A \neq \sum_{\xi, \eta}^{\oplus} p_{\xi} A p_{\eta}$, and, if the order unit space $\sum_{\xi, \eta}^{\oplus} p_{\xi} A p_{\eta}$ is not weakly closed then $\sum_{\xi, \eta}^{\oplus} p_{\xi} A p_{\eta}$ is not a C*-algebra.

INTRODUCTION

In the given article the notion of infinite order decomposition of a C*-algebra with respect to an infinite orthogonal set of projections is investigated. It is known that for any projection p of a C*-algebra A the next equality is valid $A = p A p \oplus p A (1 - p) \oplus (1 - p) A p \oplus (1 - p) A (1 - p)$, where \oplus is a direct sum of spaces. In the given article we investigated an infinite analog of this decomposition, an infinite order decomposition. The notion of infinite order decomposition was introduced in [AFN]. The next theorems belong to [AFN]:

let A be a C*-algebra of on a Hilbert space H , $\{p_{\xi}\}$ be an infinite orthogonal set of projections in A with the least upper bound 1 in the algebra $B(H)$. Then

- 1) if the order unit space $\sum_{\xi, \eta}^{\oplus} p_{\xi} A p_{\eta}$ is monotone complete in $B(H)$ (i.e. ultra-weakly closed), then $\sum_{\xi, \eta}^{\oplus} p_{\xi} A p_{\eta}$ is a C*-algebra.
- 2) if A is monotone complete in $B(H)$ (i.e. a von Neumann algebra), then $A = \sum_{\xi, \eta}^{\oplus} p_{\xi} A p_{\eta}$.

In the given article we proved that for the infinite order decomposition $\sum_{\xi, \eta}^{\oplus} p_{\xi} A p_{\eta}$ of a C*-algebra A with respect to an infinite orthogonal set $\{p_i\}$ of projections of A , if $p_{\xi} A p_{\xi}$ is a von Neumann algebra for any ξ then $\sum_{\xi, \eta}^{\oplus} p_{\xi} A p_{\eta}$ is a von Neumann algebra. For this propose it was constructed a multiplication and an involution corresponding to infinite order decompositions. It turns out, the order and the norm defined in the infinite order decomposition of a C*-algebra on a Hilbert space H coincide with the usual order and the norm in the algebra $B(H)$. Also, it is proved that, if a C*-algebra A with an infinite orthogonal set $\{p_{\xi}\}$ of projections in A such that $\sup_{\xi} p_{\xi} = 1$ is not a von Neumann algebra, projections of the set $\{p_{\xi}\}$ are pairwise equivalent then $A \neq \sum_{\xi, \eta}^{\oplus} p_{\xi} A p_{\eta}$. Moreover if the order unit space $\sum_{\xi, \eta}^{\oplus} p_{\xi} A p_{\eta}$ is not weakly closed then $\sum_{\xi, \eta}^{\oplus} p_{\xi} A p_{\eta}$ is not a C*-algebra.

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1. INFINITE ORDER DECOMPOSITIONS

Let A be a C^* -algebra on a Hilbert space H , $\{p_\xi\}$ be an infinite orthogonal set of projections of the algebra A with the least upper bound 1 in the algebra A . By $\sum_{\xi,\eta}^\oplus p_\xi A p_\eta$ we denote the set

$$\{\{a_{\xi,\eta}\} : a_{\xi,\eta} \in p_\xi A p_\eta \text{ for all } \xi, \eta, \text{ and there exists such number}$$

$$K \in \mathbf{R} \text{ that } \left\| \sum_{k,l=1}^n a_{kl} \right\| \leq K \text{ for all } n \in N \text{ and } \{a_{kl}\}_{k,l=1}^n \subseteq \{a_{\xi,\eta}\}\},$$

and say $\sum_{\xi,\eta}^\oplus p_\xi A p_\eta$ is an *infinite order decomposition* of the algebra A .

Let A be a C^* -algebra on a Hilbert space H , $\{p_\xi\}$ be an infinite orthogonal set of projections of the algebra A with the least upper bound 1 in the algebra $B(H)$. We define a relation of an order \leq in the vector space $\sum_{\xi,\eta}^\oplus p_\xi A p_\eta$ as follows: for elements $\{a_{\xi,\eta}\}, \{b_{\xi,\eta}\} \in \sum_{\xi,\eta}^\oplus p_\xi A p_\eta$, if for all $n \in N$, $\{p_k\}_{k=1}^n \subset \{p_\xi\}$ the inequality $\sum_{k,l=1}^n a_{kl} \leq \sum_{k,l=1}^n b_{kl}$ holds, then we will write $\{a_{\xi,\eta}\} \leq \{b_{\xi,\eta}\}$. Also, the map $\{a_{\xi,\eta}\} \rightarrow \|\{a_{\xi,\eta}\}\|$, $\{a_{\xi,\eta}\} \in \sum_{\xi,\eta}^\oplus p_\xi A p_\eta$, where $\|\{a_{\xi,\eta}\}\| = \sup\{\|\sum_{k,l=1}^n a_{kl}\| : n \in N, \{a_{kl}\}_{k,l=1}^n \subseteq \{a_{\xi,\eta}\}\}$, is a norm on vector space $\sum_{\xi,\eta}^\oplus p_\xi A p_\eta$.

Example. Let n be an arbitrary infinite cardinal number, Ξ be a set of indexes of the cardinality n . Let $\{e_{ij}\}$ be a set of matrix units such that e_{ij} is a $n \times n$ -dimensional matrix, i.e. $e_{ij} = (a_{\alpha\beta})_{\alpha\beta \in \Xi}$, the (i, j) -th component of which is 1, i.e. $a_{ij} = 1$, and the rest components are zeros. Let $\{m_\xi\}_{\xi \in \Xi}$ be a set of $n \times n$ -dimensional matrixes. By $\sum_{\xi \in \Xi} m_\xi$ we denote the matrix whose components are sums of the corresponding components of matrixes of the set $\{m_\xi\}_{\xi \in \Xi}$. Let

$$M_n(\mathbf{C}) = \{\{\lambda_{ij} e_{ij}\} : \text{for all indexes } i, j \lambda_{ij} \in \mathbf{C},$$

and there exists such number $K \in \mathbf{R}$, that for all $n \in N$

$$\text{and } \{e_{kl}\}_{k,l=1}^n \subseteq \{e_{ij}\} \left\| \sum_{k,l=1}^n \lambda_{kl} e_{kl} \right\| \leq K\},$$

where $\|\cdot\|$ is a norm of a matrix. It is easy to see that $M_n(\mathbf{C})$ is a vector space. The set $M_n(\mathbf{C})$, defined above, coincides with the next set:

$$\mathcal{M}_n(\mathbf{C}) = \{\{\lambda_{ij} e_{ij}\} : \text{for all indexes } i, j \lambda_{ij} \in \mathbf{C},$$

and there exists such number $K \in \mathbf{R}$ that for all

$$\{x_i\} \in l_2(\Xi) \text{ the next inequality holds } \sum_{j \in \Xi} \left| \sum_{i \in \Xi} \lambda_{ij} x_i \right|^2 \leq K^2 \sum_{i \in \Xi} |x_i|^2\},$$

where $l_2(\Xi)$ is a Hilbert space on \mathbf{C} with elements $\{x_i\}_{i \in \Xi}$, where $x_i \in \mathbf{C}$ for all $i \in \Xi$.

The associative multiplication of elements in $M_n(\mathbf{C})$ can be defined as follows: if $x = \sum_{ij \in \Xi} \lambda_{ij} e_{ij}$, $y = \sum_{ij \in \Xi} \mu_{ij} e_{ij}$ are elements of $M_n(\mathbf{C})$ then $xy = \sum_{ij \in \Xi} \sum_{\xi \in \Xi} \lambda_{i\xi} \mu_{\xi j} e_{ij}$. On this operation $M_n(\mathbf{C})$ is an associative algebra and $M_n(\mathbf{C}) = B(l_2(\Xi))$, where $B(l_2(\Xi))$ is the associative algebra of all bounded linear operators on the Hilbert space $l_2(\Xi)$. Then $M_n(\mathbf{C})$ is a von Neumann algebra of infinite $n \times n$ -dimensional matrixes on \mathbf{C} , who is defined by its own infinite order decomposition.

Analogously, if we take the algebra $B(H)$ of all bounded linear operators on an arbitrary Hilbert space H and if $\{q_i\}$ is an arbitrary maximal orthogonal set of minimal projections of the algebra $B(H)$, then $B(H) = \sum_{ij}^{\oplus} q_i B(H) q_j$ (see [AFN]).

Let A be a C^* -algebra, $\{p_i\}$ be an infinite orthogonal set of projections with the least upper bound 1 in the algebra A and let $\mathcal{A} = \{\{p_i a p_j\} : a \in A\}$. Then $A \equiv \mathcal{A}$ (see [AFN2]).

Lemma 1. *Let A be a C^* -algebra, $\{p_\xi\}$ be an infinite orthogonal set of projections of the algebra A with the least upper bound 1 in the algebra A . Then, $\sum_{\xi, \eta}^{\oplus} p_\xi A p_\eta$ is a vector space with the next componentwise algebraic operations*

$$\lambda \cdot \{a_{\xi\eta}\} = \{\lambda a_{\xi\eta}\}, \lambda \in \mathbf{C}$$

$$\{a_{\xi\eta}\} + \{b_{\xi\eta}\} = \{a_{\xi\eta} + b_{\xi\eta}\}, a_{\xi\eta}, b_{\xi\eta} \in \sum_{\xi, \eta}^{\oplus} p_\xi A p_\eta.$$

And the space \mathcal{A} is a vector subspace of the vector space $\sum_{\xi, \eta}^{\oplus} p_\xi A p_\eta$.

Lemma 2. *Let A be a C^* -algebra, $\{p_\xi\}$ be an infinite orthogonal set of projections of the algebra A with the least upper bound 1 in the algebra A . Then, the map $\{a_{\xi, \eta}\} \rightarrow \|\{a_{\xi, \eta}\}\|$, $\{a_{\xi, \eta}\} \in \sum_{\xi, \eta}^{\oplus} p_\xi A p_\eta$, where $\|\{a_{\xi, \eta}\}\| = \sup\{\|\sum_{kl=1}^n a_{kl}\| : n \in \mathbf{N}, \{a_{kl}\}_{kl=1}^n \subseteq \{a_{\xi, \eta}\}\}$, is a norm, and $\sum_{\xi, \eta}^{\oplus} p_\xi A p_\eta$ is a Banach space with this norm.*

Proof. It is clear, that for any element $\{a_{\xi, \eta}\} \in \sum_{\xi, \eta}^{\oplus} p_\xi A p_\eta$, if $\|\{a_{\xi, \eta}\}\| = 0$, then $a_{\xi, \eta} = 0$ for all ξ, η , i.e. $\{a_{\xi, \eta}\} = 0$. The rest conditions in the definition of the norm also can be easily checked.

Let (a_n) be an arbitrary Cauchy sequence in the space $\sum_{\xi, \eta}^{\oplus} p_\xi A p_\eta$, i.e. for any positive number $\varepsilon > 0$ there exists $n \in \mathbf{N}$ such, that for all $n_1 \geq n$, $n_2 \geq n$ the inequality $\|a_{n_1} - a_{n_2}\| < \varepsilon$ holds. Then the set $\{\|a_n\|\}$ is bounded by some number $K \in \mathbf{R}_+$ and for any finite set $\{p_k\}_{k=1}^n \subset \{p_i\}$ the sequence $(p a_n p)$ is a Cauchy sequence, where $p = \sum_{k=1}^n p_k$. Then, since A is a Banach space, then $\lim_{n \rightarrow \infty} p a_n p \in A$.

Let $a_{\xi, \eta} = \lim_{n \rightarrow \infty} p_\xi a_n p_\eta$ for all ξ and η . Then $\|\sum_{kl=1}^n a_{kl}\| \leq K$ for all $n \in \mathbf{N}$ and $\{a_{kl}\}_{kl=1}^n \subseteq \{a_{\xi, \eta}\}$. Hence $\{a_{\xi, \eta}\} \in \sum_{\xi, \eta}^{\oplus} p_\xi A p_\eta$. \square

The definition of the order in $\sum_{\xi, \eta}^{\oplus} p_\xi A p_\eta$ is equivalent to the next condition: for the elements $\{a_{\xi\eta}\}, \{b_{\xi\eta}\} \in \sum_{\xi, \eta}^{\oplus} p_\xi A p_\eta$, if for all $n \in \mathbf{N}$ and $\{p_k\}_{k=1}^n \subseteq \{p_i\}$ the equality $\{a_{kl}\}_{k,l=1}^n \leq \{b_{kl}\}_{k,l=1}^n$ holds in the algebra \mathcal{A} , then $\{a_{\xi\eta}\} \leq \{b_{\xi\eta}\}$.

Proposition 3. *Let A be a C^* -algebra on a Hilbert space H , $\{p_\xi\}$ be an infinite orthogonal set of projections in A with the least upper bound 1 in the algebra $B(H)$. Then the relation \leq , introduced above, is a relation of a partial order, and the space $\sum_{\xi, \eta}^{\oplus} p_\xi A p_\eta$ is an order unit space with this order. In this case $\mathcal{A} = \{\{p_\xi a p_\eta\} : a \in A\}$ is an order unit subspace of the order unit space $\sum_{\xi, \eta}^{\oplus} p_\xi A p_\eta$.*

Proof. Let $\mathcal{M} = \sum_{\xi, \eta}^{\oplus} p_\xi A p_\eta$. The space \mathcal{M} is a partially ordered vector space, i.e. $\mathcal{M}_+ \cap \mathcal{M}_- = \{0\}$, where $\mathcal{M}_+ = \{\{a_{\xi\eta}\} \in \mathcal{M} : \{a_{\xi\eta}\} \geq 0\}$, $\mathcal{M}_- = \{\{a_{\xi\eta}\} \in \mathcal{M} : \{a_{\xi\eta}\} \leq 0\}$.

By the definition of the order the partially ordered vector space \mathcal{M} is Archimedean. Let $\{a_{\xi\eta}\} \in \mathcal{M}$. Since for any finite set $\{p_k\}_{k=1}^n \subset \{p_\xi\}$ the inequality $-\|\{a_{\xi, \eta}\}\| p \leq p \{a_{\xi, \eta}\} p \leq \|\{a_{\xi, \eta}\}\| p$ holds, where $p = \sum_{k=1}^n p_k$, then by the definition of the order

$-\|\{a_{\xi,\eta}\}\|1 \leq \{a_{\xi,\eta}\} \leq \|\{a_{\xi,\eta}\}\|1$, and the unit of A is an order unit of the partially ordered vector space \mathcal{M} . Thus, \mathcal{M} is an order unit space.

By lemma 1 \mathcal{A} is an order unit subspace of the order unit space \mathcal{M} . \triangleright

Proposition 4. *Let A be a C^* -algebra on a Hilbert space H , $\{p_i\}$ be an infinite orthogonal set of projections in A with the least upper bound 1 in the algebra $B(H)$. Then the order unit space $\mathcal{A} = \{\{p_\xi a p_\eta\} : a \in A\}$ is a C^* -algebra, where the operation of multiplication of \mathcal{A} defines as follows*

$$\cdot : \langle \{p_\xi a p_\eta\}, \{p_\xi b p_\eta\} \rangle \rightarrow \{p_\xi a b p_\eta\}, \{p_\xi a p_\eta\}, \{p_\xi b p_\eta\} \in \mathcal{A}.$$

Proof. By lemma 4 in [AFN2] the map

$$\mathcal{I} : a \in A \rightarrow \{p_\xi a p_\eta\} \in \mathcal{A}$$

is a one-to-one map. In this case

$$\mathcal{I}(a)\mathcal{I}(b) = \mathcal{I}(ab)$$

by the definition in proposition 4 of the multiplication, and $\mathcal{I}(a) = \{p_\xi a p_\eta\}$, $\mathcal{I}(b) = \{p_\xi b p_\eta\}$, $\mathcal{I}(ab) = \{p_\xi a b p_\eta\}$. Hence, the operation, introduced in the formulation of proposition 4 is an associative multiplication and the map \mathcal{I} is an isomorphism of the algebras A and \mathcal{A} .

By proposition 3 the isomorphism \mathcal{I} is isometrical. Therefore \mathcal{A} is a C^* -algebra with this operation. \triangleright

Example 1. We take the algebra $B(H)$ of all bounded linear operators on a Hilbert space H . Let $\{q_i\}$ be a maximal orthogonal set of minimal projections of the algebra $B(H)$. Then $\sup_i q_i = 1$ and by lemma 4 in [AFN2] and proposition 4 the space $\mathcal{B}(\mathcal{H}) = \{\{q_i a q_j\} : a \in B(H)\}$ can be identified with $B(H)$ as C^* -algebras in the sense of the map

$$\mathcal{I} : a \in B(H) \rightarrow \{q_i a q_j\} \in \mathcal{B}(\mathcal{H}).$$

In this case the operation associative multiplication in $\mathcal{B}(\mathcal{H})$ is defined as follows

$$\cdot : \langle \{q_i a q_j\}, \{q_i b q_j\} \rangle \rightarrow \{q_i a b q_j\}, \{q_i a q_j\}, \{q_i b q_j\} \in \mathcal{B}(\mathcal{H}).$$

Let $a, b \in B(H)$, $q_i a q_j = \lambda_{ij} q_{ij}$, $q_i b q_j = \mu_{ij} q_{ij}$, where $\lambda_{ij}, \mu_{ij} \in \mathbf{C}$, $q_i = q_{ij} q_{ij}^*$, $q_j = q_{ij}^* q_{ij}$, for all indexes i and j . Then this multiplication coincides with the next bilinear operation

$$\cdot : \langle \{q_i a q_j\}, \{q_i b q_j\} \rangle \rightarrow \left\{ \sum_{\xi} \lambda_{i\xi} \mu_{\xi j} q_{ij} \right\}, \{q_i a q_j\}, \{q_i b q_j\} \in \mathcal{B}(\mathcal{H}).$$

Remark 1. Let A be a C^* -algebra on a Hilbert space H , $\{p_i\}$ be an infinite orthogonal set of projections in A with the least upper bound 1 in the algebra $B(H)$. Then by proposition 4 $\mathcal{A} = \{\{p_\xi a p_\eta\} : a \in A\}$ is a C^* -algebra. In this case the involution on the algebra \mathcal{A} coincides with the next map:

$$\{p_\xi a p_\eta\}^* = \{p_\xi a^* p_\eta\}, \quad a \in A.$$

Indeed, the identification $\mathcal{A} \equiv A$ gives us $a = \{p_\xi a p_\eta\}$ and $a^* = \{p_\xi a^* p_\eta\}$ for all $a \in A$. Then $\{p_\xi a p_\eta\}^* = a^* = \{p_\xi a^* p_\eta\}$ for any $a \in A$. Let $\mathcal{A}_{sa} = \{\{p_\xi a p_\eta\} : a \in \mathcal{A}_{sa}\}$. Then $\mathcal{A} = \mathcal{A}_{sa} + i\mathcal{A}_{sa}$. Indeed, $\{p_\xi a p_\eta\}^* = a^* = a = \{p_\xi a p_\eta\}$ for any $a \in \mathcal{A}_{sa}$.

Let $\mathcal{N} = \{\{p_\xi a p_\eta\} : a \in B(H)\}$. By lemma 4 in [AFN2] and by proposition 4 $\mathcal{N} \equiv B(H)$. Therefore we will assume that $\mathcal{N} = B(H)$. Let $\mathcal{N}_{sa} = \{\{p_\xi a p_\eta\} : a \in B(H), \{p_\xi a p_\eta\}^* = \{p_\xi a p_\eta\}\}$. Then $\mathcal{N} = \mathcal{N}_{sa} + i\mathcal{N}_{sa}$. Note that $\{p_\xi a p_\eta\}^* = \{p_\xi a p_\eta\}$ if and only if $(p_\xi a p_\eta)^* = p_\eta a p_\xi$ for all ξ, η .

Lemma 5. *Let $B(H)$ be the algebra of all bounded linear operators on a Hilbert space H . Let $\{p_\xi\}$ be an infinite orthogonal set of projections of the algebra $B(H)$ with the least upper bound 1. Then the associative multiplication of the algebra \mathcal{N} (hence of the algebra $B(H)$) coincides with the next operation*

$$\{p_\xi a p_\eta\} \star \{p_\xi b p_\eta\} = \left\{ \sum_i p_\xi a p_i p_i b p_\eta \right\}, \{p_\xi a p_\eta\}, \{p_\xi b p_\eta\} \in \mathcal{N}$$

where the sum \sum in the right part of the equality is an ultraweak limit of the net of finite sums of elements in the set $\{p_\xi a p_i p_i b p_\eta\}_{\xi, \eta}$.

Proof. Let $\{p_k\}_{k=1}^n$ be a finite subset of the set $\{p_\xi\}$. Note that $\sup_i p_i = 1$ in the algebra $B(H)$, i.e. the net of all finite sums of the kind $\sum_{k=1}^n p_k$ of orthogonal projections of the set $\{p_\xi\}$ ultraweakly converges to the identity operator in $B(H)$. By the ultraweakly continuity of the operator of multiplication $T(b) = ab, b \in B(H)$, where $a \in B(H)$, the net of finite sums of elements in the set $\{p_\xi a p_i p_i b p_\eta\}_{\xi, \eta}$ ultraweakly converges in $B(H)$ and $\sum_i p_\xi a p_i p_i b p_\eta = p_\xi a b p_\eta$ for all ξ, η . Hence the operation of multiplication \star of the algebra \mathcal{N} coincides with the operation, introduced in proposition 4. And the operation of the associative multiplication, introduced in proposition 4 coincides with the multiplication in the algebra $B(H)$ in the sense $\mathcal{N} \equiv B(H)$. \triangleright

Proposition 6. *Let A be a C^* -algebra on a Hilbert space H , $\{p_\xi\}$ be an infinite orthogonal set of projections in A with the least upper bound 1 in the algebra $B(H)$. Then the operation of associative multiplication of the algebra \mathcal{A} coincides with the inducing on \mathcal{N} of the operation, defined in lemma 5.*

Proof. Let $\{p_\xi a p_\eta\}, \{p_\xi b p_\eta\}$ be elements of \mathcal{A}_{sa} and $\{p_k\}_{k=1}^n$ be a finite subset of the set $\{p_\xi\}$ and $p = \sum_{k=1}^n p_k$. We have the net of all finite sums of the kind $\sum_{k=1}^n p_k$ of orthogonal projections of the set $\{p_\xi\}$ ultraweakly converges to the identity operator in $B(H)$. Then for all ξ, η the element $\{p_\xi a b p_\eta\}$ is an ultraweak limit in $B(H)$ of the net $\{\sum_i p_\xi a p_i p_i b p_\eta\}$ of all finite sums $\{\sum_{k=1}^n p_\xi a p_k p_k b p_\eta\}$ on all subsets $\{p_k\}_{k=1}^n \subset \{p_\xi\}$, and the element $\{p_\xi a b p_\eta\}$ belongs to \mathcal{A} . Hence the assertion of proposition 6 holds. \triangleright

Remark 2. Let A be a C^* -algebra on a Hilbert space H , $\{p_i\}$ be an infinite orthogonal set of projections in A with the least upper bound 1 in the algebra $B(H)$. Note that then by lemma 4 in [AFN2] the order and the norm in the vector space $\sum_{i,j}^\oplus p_i A p_j$ can be introduced as follows: we write $\{a_{ij}\} \geq 0$, if this element is zero or positive element in $B(H)$ in the sense of the equality $B(H) = \sum_{\xi, \eta}^\oplus q_\xi B(H) q_\eta$, where $\{q_\xi\}$ is an arbitrary maximal orthogonal set of minimal projections of the algebra $B(H)$; $\|\{a_{ij}\}\|$ is equal to the norm in $B(H)$ of this element in the sense of the equality $B(H) = \sum_{\xi, \eta}^\oplus q_\xi B(H) q_\eta$ (example 1). By lemmas 3 and 4 in [AFN2] they coincide with the order and the norm defined in lemma 2 and proposition 3, correspondingly.

Remark 3. Suppose that all conditions of remark 2 hold. Then $B(H) \equiv \mathcal{B}(\mathcal{H}) = \sum_{\xi, \eta}^\oplus q_\xi B(H) q_\eta$, where $\mathcal{B}(\mathcal{H}) = \{\{q_\xi a q_\eta\} : a \in B(H)\}$. Also, we have $\sum_{i,j}^\oplus p_i A p_j$ is a Banach space and an order unit space (lemma 2, Proposition 3). Suppose that

$\{q_\xi\}$ is such maximal orthogonal set of minimal projections of the algebra $B(H)$ that $p_i = \sup_\eta q_\eta$, for some subset $\{q_\eta\} \subset \{q_\xi\}$, for all i . Note that $B(H) \equiv \{\{p_i a p_j\} : a \in B(H)\} = \sum_{ij}^\oplus p_i B(H) p_j$. By propositions 4 and 6 the order unit space $\mathcal{A} = \{\{p_i a p_j\} : a \in A\}$ is closed concerning the associative multiplication of the algebra $\sum_{ij}^\oplus p_i B(H) p_j$ (what is the same that $\mathcal{N} = \{\{p_i a p_j\} : a \in B(H)\}$).

At the same time, the order unit space $\sum_{ij}^\oplus p_i A p_j$ is the order unit subspace of the algebra $\sum_{ij}^\oplus p_i B(H) p_j$.

Since $B(H) \equiv \sum_{ij}^\oplus p_i B(H) p_j$, then $\sum_{ij}^\oplus p_i B(H) p_j$ is a von Neumann algebra, and without loss of generality, this algebra can be considered as the algebra $B(H)$.

Note that if the space $\sum_{ij}^\oplus p_i A p_j$ is closed concerning the associative multiplication of the algebra $\sum_{ij}^\oplus p_i B(H) p_j$, then $\sum_{ij}^\oplus p_i A p_j$ is a C^* -algebra. Also, when we consider the C^* -algebra A with the conditions which are listed above, then we have the algebra $\sum_{ij}^\oplus p_i B(H) p_j$ (i.e. actually the algebra $B(H)$) and the vector space $\sum_{ij}^\oplus p_i A p_j$ as an order unit subspace of the algebra $\sum_{ij}^\oplus p_i B(H) p_j$. Then we have

$$\mathcal{A} \subseteq \sum_{ij}^\oplus p_i A p_j \subseteq \sum_{ij}^\oplus p_i B(H) p_j.$$

Thus, further, when we say that $\sum_{ij}^\oplus p_i A p_j$ is a C^* -algebra we assume that the vector space $\sum_{ij}^\oplus p_i A p_j$ is closed concerning the associative multiplication of the algebra $\sum_{ij}^\oplus p_i B(H) p_j$.

The involution in the sense of the identification $\sum_{ij}^\oplus p_i B(H) p_j \equiv B(H)$ coincides with the next map:

$$\{a_{ij}\}^* = \{a_{ji}^*\}, \{a_{ij}\} \in \sum_{ij}^\oplus p_i B(H) p_j.$$

Indeed, there exists an element $a \in B(H)$ such that $a = \{a_{ij}\} = \{p_i a p_j\}$. Then $a^* = \{p_i a^* p_j\}$ in the sense of $B(H) \equiv \mathcal{N}$. We have $a_{ij} = p_i a p_j$, $a_{ij}^* = p_j a^* p_i$ for all i, j . Therefore $\{p_i a^* p_j\} = \{a_{ji}^*\}$. Hence $a^* = \{a_{ji}^*\}$. Let $(\sum_{ij}^\oplus p_i B(H) p_j)_{sa} = \{\{a_{ij}\} : \{a_{ij}\} \in \sum_{ij}^\oplus p_i B(H) p_j, \{a_{ij}\}^* = \{a_{ij}\}\}$. Then

$$\sum_{ij}^\oplus p_i B(H) p_j = (\sum_{ij}^\oplus p_i B(H) p_j)_{sa} + i(\sum_{ij}^\oplus p_i B(H) p_j)_{sa}.$$

Lemma 7. *Let A be a C^* -algebra on a Hilbert space H , $\{p_i\}$ be an infinite orthogonal set of projections of the algebra A with least upper bound 1 in $B(H)$ and $(\sum_{ij}^\oplus p_i A p_j)_{sa} = \{\{a_{ij}\} : \{a_{ij}\} \in \sum_{ij}^\oplus p_i A p_j, \{a_{ij}\}^* = \{a_{ij}\}\}$. Then*

$$\sum_{ij}^\oplus p_i A p_j = (\sum_{ij}^\oplus p_i A p_j)_{sa} + i(\sum_{ij}^\oplus p_i A p_j)_{sa}. \quad (**)$$

In this case the equality $\{a_{ij}\}^ = \{a_{ij}\}$ holds for $\{a_{ij}\} \in \sum_{ij}^\oplus p_i A p_j$ if and only if $a_{ij}^* = a_{ji}$ for all i, j .*

Proof. Let $\{a_{ij}\} \in \sum_{ij}^\oplus p_i A p_j$. We have $a_{ij} + a_{ji} = a_1 + i a_2$, where $a_1, a_2 \in (\sum_{ij}^\oplus p_i A p_j)_{sa}$, for all i and j , since $a_{ij} + a_{ji} \in A$. Then $a_{ij} + a_{ji} = p_i a_1 p_j +$

$p_j a_1 p_i + i(p_i a_2 p_j + p_j a_2 p_i)$, $a_1 = p_i a_1 p_j + p_j a_1 p_i$, $a_2 = p_i a_2 p_j + p_j a_2 p_i$ for all i and j . Let $a_{ij}^1 = p_i a_1 p_j + p_j a_1 p_i$, $a_{ij}^2 = p_i a_2 p_j + p_j a_2 p_i$ for all i and j . Then by the definition of the vector space $\sum_{ij}^{\oplus} p_i A p_j$ we have $\{a_{ij}^1\}, \{a_{ij}^2\} \in \sum_{ij}^{\oplus} p_i A p_j$. In this case $\{a_{ij}^k\}^* = \{a_{ij}^k\}$, $k = 1, 2$. Since the element $\{a_{ij}\} \in \sum_{ij}^{\oplus} p_i A p_j$ was chosen arbitrarily we have the equality (**).

The rest part of the assertion of lemma 7 holds by the definition of the self-adjoint elements $\{a_{ij}^k\}$, $k = 1, 2$. \triangleright

Lemma 8. *Let $B(H)$ be a $*$ -algebra of all bounded linear operators on a Hilbert space H , $\{p_\xi\}$ be an infinite orthogonal set of projections of $B(H)$ with the least upper bound 1. Then the associative multiplication of the algebra $\sum_{\xi, \eta}^{\oplus} p_\xi B(H) p_\eta$ (i.e. of the algebra $B(H)$) coincides with the associative multiplication defined as follows:*

$$\cdot : \langle \{a_{\xi, \eta}\}, \{b_{\xi, \eta}\} \rangle \rightarrow \left\{ \sum_i a_{\xi i} b_{i \eta}, \{a_{\xi \eta}\}, \{b_{\xi \eta}\} \in \left(\sum_{\xi, \eta}^{\oplus} p_\xi B(H) p_\eta \right) \right\}.$$

Proof. Let $\{a_{\xi \eta}\}, \{b_{\xi \eta}\} \in (\sum_{\xi, \eta}^{\oplus} p_\xi B(H) p_\eta)$. We have $B(H) \equiv \mathcal{N} \equiv \sum_{\xi, \eta}^{\oplus} p_\xi B(H) p_\eta$. Therefore, it can be regarded that $B(H) = \mathcal{N} = \sum_{\xi, \eta}^{\oplus} p_\xi B(H) p_\eta$. There exists elements a, b in the algebra $B(H)$ such that $p_\xi a p_\eta = a_{\xi \eta}$, $p_\xi b p_\eta = b_{\xi \eta}$ for all ξ, η . Therefore $\{a_{\xi \eta}\} = \{p_\xi a p_\eta\}$, $\{b_{\xi \eta}\} = \{p_\xi b p_\eta\}$. Then by lemma 5 we have the associative multiplication of the algebra $\sum_{\xi, \eta}^{\oplus} p_\xi B(H) p_\eta$ (i.e. of the algebra $B(H)$) coincides with the operation defined in lemma 8. \triangleright

Proposition 9. ([AFN]) *Let A be a von Neumann algebra on a Hilbert space H , $\{p_i\}$ be an infinite orthogonal set of projections of the algebra A with least upper bound 1 in $B(H)$. Then $A = \sum_{\xi, \eta}^{\oplus} p_\xi A p_\eta$.*

Proof. Let a be an element of the vector space $\sum_{\xi, \eta}^{\oplus} p_\xi A p_\eta$ and $a = \{a_{\xi \eta}\}$, where $a_{\xi \xi} = p_\xi a p_\xi$, $a_{\xi \eta} = p_\xi a p_\eta$ for all ξ, η . We have $a \in B(H) = \sum_{\xi, \eta}^{\oplus} p_\xi B(H) p_\eta$ and $(\sum_{k=1}^n p_k) a (\sum_{k=1}^n p_k) \in A$ for any $\{p_k\}_{k=1}^n \subset \{p_\xi\}$. Let

$$b_n^\alpha = \sum_{kl=1}^n p_k^\alpha a p_l^\alpha = \left(\sum_{kl=1}^n p_k^\alpha \right) a \left(\sum_{kl=1}^n p_k^\alpha \right)$$

for all natural numbers n and finite subsets $\{p_k^\alpha\}_{k=1}^n \subset \{p_i\}$. Then by the proof of lemma 3 in [AFN2] the net (b_n^α) ultraweakly converges to a in $B(H)$. At the same time A is ultraweakly closed in $B(H)$. Therefore $a \in A$ and $\sum_{\xi, \eta}^{\oplus} p_\xi A p_\eta \subseteq A$. \triangleright

Lemma 10. *Let A be a C^* -algebra on a Hilbert space H , $\{p_\xi\}$ be an infinite orthogonal set of projections in A with the least upper bound 1 in the algebra $B(H)$. Then, if projections of the set $\{p_\xi\}$ are pairwise equivalent and for every index ξ the component $p_\xi A p_\xi$ is a von Neumann algebra, then the vector space $\sum_{\xi, \eta}^{\oplus} p_\xi A p_\eta$ is closed concerning the multiplication of the algebra $\sum_{\xi, \eta}^{\oplus} p_\xi B(H) p_\eta$ and $\sum_{\xi, \eta}^{\oplus} p_\xi A p_\eta$ is a C^* -algebra.*

Proof. First, note that $(p_\xi + p_\eta)A(p_\xi + p_\eta)$ is a von Neumann algebra. Indeed, for any net (a_α) in $p_\xi A p_\eta$, weakly converging in $B(H)$ the net $(a_\alpha x_{\xi \eta}^*)$ belongs to $p_\xi A p_\xi$, where $x_{\xi \eta}$ is an isometry in A such that $x_{\xi \eta} x_{\xi \eta}^* = p_\xi$, $x_{\xi \eta}^* x_{\xi \eta} = p_\eta$. Then since the net $(a_\alpha x_{\xi \eta}^*)$ weakly converges in $B(H)$ then the weak limit b in $B(H)$ of

the net $(a_\alpha x_{\xi\eta}^*)$ belongs to $p_\xi A p_\xi$. Hence $b x_{\xi\eta} \in p_\xi A p_\eta$. It is easy to see that $b x_{\xi\eta}$ is a weak limit in $B(H)$ of the net (a_α) . Hence $p_\xi A p_\eta$ is weakly closed in $B(H)$.

Let $\{a_{\xi\eta}\}, \{b_{\xi\eta}\} \in (\sum_{\xi,\eta}^\oplus p_\xi A p_\eta)$. We have

$$\sum_{\xi,\eta}^\oplus p_\xi A p_\eta \subseteq \sum_{\xi,\eta}^\oplus p_\xi B(H) p_\eta = B(H).$$

Therefore there exist elements a, b in the algebra $\sum_{\xi,\eta}^\oplus p_\xi B(H) p_\eta$ (i.e. in the algebra $B(H)$) such that $p_\xi a p_\eta = a_{\xi\eta}$, $p_\xi b p_\eta = b_{\xi\eta}$ for all ξ, η . Therefore $\{a_{\xi\eta}\} = \{p_\xi a p_\eta\}$, $\{b_{\xi\eta}\} = \{p_\xi b p_\eta\}$. We have

$$\sum_i a_{\xi_i} b_{i\eta} = p_\xi a b p_\eta$$

calculated in $\sum_{\xi,\eta}^\oplus p_\xi B(H) p_\eta$ belong to $p_\xi A p_\eta$. Since the indexes ξ, η were chosen arbitrarily and the product $\{p_\xi a p_\eta\} \{p_\xi b p_\eta\} = ab$ belongs to $\sum_{\xi,\eta}^\oplus p_\xi B(H) p_\eta$, then the product of the elements a and b belongs to $\sum_{\xi,\eta}^\oplus p_\xi A p_\eta$. Therefore the vector space $\sum_{\xi,\eta}^\oplus p_\xi A p_\eta$ is closed with respect to the associative multiplication of the algebra $\sum_{\xi,\eta}^\oplus p_\xi B(H) p_\eta$. At the same time, $\sum_{\xi,\eta}^\oplus p_\xi A p_\eta$ is a norm closed subspace of the algebra $\sum_{\xi,\eta}^\oplus p_\xi B(H) p_\eta = B(H)$. Hence $\sum_{\xi,\eta}^\oplus p_\xi A p_\eta$ is a C^* -algebra and the multiplication in $\sum_{\xi,\eta}^\oplus p_\xi A p_\eta$ can be defined as in the formulation of lemma 8. \triangleright

Theorem 11. *Let A be a C^* -algebra on a Hilbert space H , $\{p_\xi\}$ be an infinite orthogonal set of projections in A with the least upper bound 1 in the algebra $B(H)$. Suppose that projections of the set $\{p_\xi\}$ are pairwise equivalent and for any ξ $p_\xi A p_\xi$ is a von Neumann algebra. Then $\sum_{\xi,\eta}^\oplus p_\xi A p_\eta$ is a von Neumann algebra.*

Proof. Let $\{x_{\xi\eta}\}$ be such set of isometries in A that $p_\xi = x_{\xi\eta} x_{\xi\eta}^*$, $p_\eta = x_{\xi\eta}^* x_{\xi\eta}$ for all ξ, η . Let ξ, η be arbitrary indexes. We prove that $p_\xi A p_\eta$ is weakly closed. We have $p_\xi A p_\eta p_\eta A p_\xi \subseteq p_\xi A p_\xi$ and $p_\xi A p_\eta = x_{\xi\eta} A x_{\xi\eta}$. Let (a_α) be a net in $p_\xi A p_\eta$, weakly converging to an element a in $B(H)$. Then there exists a net (b_α) in $p_\xi A p_\eta$ such that $a_\alpha = x_{\xi\eta} b_\alpha x_{\xi\eta}$ for all α . By the weakly continuity of the multiplication separately on multipliers the net $(a_\alpha x_{\xi\eta}^*)$ weakly converges to the element $a x_{\xi\eta}$ in the algebra $B(H)$. Since $(a_\alpha x_{\xi\eta}^*) \subseteq p_\xi A p_\xi$ and $p_\xi A p_\xi$ is weakly closed in $B(H)$, then $a x_{\xi\eta} \in p_\xi A p_\xi$. Hence there exists an element $b \in A$ such that $a x_{\xi\eta}^* = x_{\xi\eta} b x_{\xi\eta} x_{\xi\eta}^*$. Then $a x_{\xi\eta}^* x_{\xi\eta} = x_{\xi\eta} b x_{\xi\eta} x_{\xi\eta}^* x_{\xi\eta} = x_{\xi\eta} b x_{\xi\eta} p_\eta = x_{\xi\eta} b x_{\xi\eta} \in p_\xi A p_\eta$. At the same time $a_\alpha p_\eta = a_\alpha$ for all α . Hence, $a p_\eta = a$ in the algebra $B(H)$. Since $a = a x_{\xi\eta}^* x_{\xi\eta} = x_{\xi\eta} b x_{\xi\eta} \in p_\xi A p_\eta$, then $a \in p_\xi A p_\eta$. Since the net (a_α) is chosen arbitrarily, then the component $p_\xi A p_\eta$ is weakly closed in $B(H)$. Let (a_α) be a net in $\sum_{\xi,\eta}^\oplus p_\xi A p_\eta$, weakly converging to an element a in $B(H)$. Then for all ξ and η the net $(p_\xi a_\alpha p_\eta)$ weakly converges to $p_\xi a p_\eta$ in $B(H)$. In this case, by the previous part of the proof $p_\xi a p_\eta \in p_\xi A p_\eta$ for all ξ, η . Note that $a \in \sum_{\xi,\eta}^\oplus p_\xi B(H) p_\eta$. Hence $a \in \sum_{\xi,\eta}^\oplus p_\xi A p_\eta$. Since the net (a_α) is chosen arbitrarily, then the vector space $\sum_{\xi,\eta}^\oplus p_\xi A p_\eta$ is weakly closed in the algebra $\sum_{\xi,\eta}^\oplus p_\xi B(H) p_\eta \equiv B(H)$. Therefore by lemma 10 $\sum_{\xi,\eta}^\oplus p_\xi A p_\eta$ is a von Neumann algebra. \triangleright

Proposition 12. *Let A be a monotone complete C^* -algebra on a Hilbert space H , $\{p_\xi\}$ be an infinite orthogonal set of projections in A with the least upper bound 1 in the algebra $B(H)$. Then the order unit space $\sum_{\xi,\eta}^\oplus p_\xi A p_\eta$ is monotone complete.*

Proof. We have the C^* -subalgebra $p_\xi A p_\xi$ is monotone complete for any index ξ . Let $\{p_k\}_{k=1}^n$ be a finite subset of the set $\{p_\xi\}$ and $p = \sum_{k=1}^n p_k$. Then the C^* -subalgebra pAp is also monotone complete.

Let (a_α) be a bounded monotone increasing net in $\sum_{\xi, \eta}^\oplus p_\xi A p_\eta$. Since for any finite subset $\{p_k\}_{k=1}^n \subseteq \{p_\xi\}$ the subalgebra $(\sum_{k=1}^n p_k)A(\sum_{k=1}^n p_k)$ is monotone complete then

$$\sup_\alpha \left(\sum_{k=1}^n p_k \right) a_\alpha \left(\sum_{k=1}^n p_k \right) \in \left(\sum_{k=1}^n p_k \right) A \left(\sum_{k=1}^n p_k \right).$$

Hence, $\{a_{\xi\eta}\} = \{\sup_\alpha p_\xi a_\alpha p_\xi\} \cup \{p_\xi (\sup_\alpha (p_\xi + p_\eta) a_\alpha (p_\xi + p_\eta)) p_\eta\}_{\xi \neq \eta}$ is an element of the order unit space $\sum_{\xi, \eta}^\oplus p_\xi A p_\eta$. It can be checked straightforwardly using the definition of the order in the order unit space $\sum_{\xi, \eta}^\oplus p_\xi A p_\eta$ that the element $\{a_{\xi\eta}\}$ is the least upper bound of the net (a_α) . Since the net (a_α) was chosen arbitrarily then the order unit space $\sum_{\xi, \eta}^\oplus p_\xi A p_\eta$ is monotone complete. \triangleright

Theorem 13. *Let A be a monotone complete C^* -algebra of bounded linear operators on a Hilbert space H , $\{p_\xi\}$ be an infinite orthogonal set of projections in A with the least upper bound 1 in the algebra $B(H)$. Suppose that projections of the set $\{p_\xi\}$ are pairwise equivalent and A is not a von Neumann algebra. Then $A \neq \sum_{\xi, \eta}^\oplus p_\xi A p_\eta$ (i.e. $\mathcal{A} := \{\{p_\xi a p_\eta\} : a \in A\} \neq \sum_{\xi, \eta}^\oplus p_\xi A p_\eta$).*

Proof. We have there exists a bounded monotone increasing net (a_α) of elements in A , the least upper bound $\sup_A a_\alpha$ in the algebra A and the least upper bound $\sup_{\sum_{\xi, \eta}^\oplus p_\xi B(H) p_\eta} a_\alpha$ in the algebra $\sum_{\xi, \eta}^\oplus p_\xi B(H) p_\eta$ of which are different. Otherwise A is a von Neumann algebra.

By the definition of the order in the algebra $\sum_{\xi, \eta}^\oplus p_\xi B(H) p_\eta$ there exists a projection $p \in \{p_\xi\}$ such that the least upper bound $\sup_{pAp} p a_\alpha p$ in the algebra pAp and the least upper bound $\sup_{pB(H)p} p a_\alpha p$ in the algebra $pB(H)p$ of the bounded monotone increasing net $(p a_\alpha p)$ of elements in pAp are different. Indeed, let $a = \sup_A a_\alpha$, $b = \sup_{\sum_{\xi, \eta}^\oplus p_\xi B(H) p_\eta} a_\alpha$. Since $A \subseteq \sum_{\xi, \eta}^\oplus p_\xi B(H) p_\eta$, then $b \leq a$ and $0 \leq a - b$. Hence, if $p_\xi(a - b)p_\xi = 0$ for all ξ , then $p_\xi(a - b) = (a - b)p_\xi = 0$. Therefore by lemma 2 in [AFN2] $a - b = 0$, i.e. $a = b$. Hence pAp is not a von Neumann algebra.

We have there exists an infinite orthogonal set $\{e_i\}$ of projections in pAp , the least upper bound $\sup_{pAp} e_i$ in the algebra pAp and the least upper bound $\sup_{pB(H)p} e_i$ in the algebra $pB(H)p$ of which are different. Otherwise pAp is a von Neumann algebra.

Indeed, any maximal commutative subalgebra A_o of pAp is monotone complete. For any normal positive linear functional $\rho \in B(H)$ and for any infinite orthogonal set $\{q_i\}$ of projections in A_o we have $\rho(\sup_i q_i) = \sum_i \rho(q_i)$, where $\sup_i q_i$ is the least upper bound of the set $\{q_i\}$ in A_o . Hence by the theorem on extension of a σ -additive measure to a normal linear functional $\rho|_{A_o}$ is a normal functional on A_o . Hence A_o is a commutative von Neumann algebra. At the same time the maximal commutative subalgebra A_o of the algebra $\sum_{\xi, \eta}^\oplus p_\xi A p_\eta$ is chosen arbitrarily. Therefore by [GKP] $\sum_{\xi, \eta}^\oplus p_\xi A p_\eta$ is a von Neumann algebra. What is impossible.

Let $\{x_{\xi\eta}\}$ be such set of isometries in A that $p_\xi = x_{\xi\eta} x_{\xi\eta}^*$, $p_\eta = x_{\xi\eta}^* x_{\xi\eta}$ for all ξ, η , and let $p_1 = p$. Let $\{x_{1\xi}\}$ be the subset of the set $\{x_{\xi\eta}\}$ such that $p_1 = x_{1\xi} x_{1\xi}^*$, $p_\xi = x_{1\xi}^* x_{1\xi}$ for all ξ . Without loss of generality we regard that the set of indexes i for $\{e_i\}$ is a subset of the set of indexes ξ for $\{p_\xi\}$. Let $\{e_i x_{1i}\}$ be a set of

all components of some infinite dimensional matrix $\{a_{\xi\eta}\}$, the components, which are does not present, are zeros and $\{x_{1i}^*e_i^*\}$ be also an analogous matrix, which coincides with $\{a_{\xi\eta}^*\}$. We have $\sum_i e_i x_{1i} x_{1i}^* e_i^* = \sum_i e_i p_1 e_i^* = \sum_i e_i e_i^* = \sum_i e_i \leq \sup_{pAp} e_i$. Therefore $\{a_{\xi\eta}\} \in \sum_{\xi,\eta}^\oplus p_\xi A p_\eta$. Then $\{a_{\xi\eta}^*\} \in \sum_{\xi,\eta}^\oplus p_\xi A p_\eta$. Therefore if $\{a_{\xi\eta}\} \in A$ (i.e. in $\mathcal{A} := \{\{p_\xi a p_\eta\} : a \in A\}$) then the product $\{a_{\xi\eta}\} \cdot \{a_{\xi\eta}^*\}$ in $\sum_{ij}^\oplus p_i B(H) p_j$ belongs to $\sum_{\xi,\eta}^\oplus p_\xi A p_\eta$. In this case we have the infinite dimensional matrix $\{a_{\xi\eta}\} \cdot \{a_{\xi\eta}^*\}$ contains the component $\sum_i e_i x_{1i} \cdot x_{1i}^* e_i^*$ such that $\sum_i e_i x_{1i} \cdot x_{1i}^* e_i^* = p_1 (\sum_i e_i x_{1i} \cdot x_{1i}^* e_i^*) p_1$. Consequently, $p_1(\{a_{\xi\eta}\} \cdot \{a_{\xi\eta}^*\}) p_1 = \sum_i e_i x_{1i} \cdot x_{1i}^* e_i^*$. Hence $\sum_i e_i x_{1i} \cdot x_{1i}^* e_i^* \in p_1 (\sum_{\xi,\eta}^\oplus p_\xi A p_\eta) p_1 = p_1 A p_1$. Since $\sum_i e_i x_{1i} \cdot x_{1i}^* e_i^* = \sum_i e_i p_1 e_i^* = \sum_i e_i e_i^* = \sum_i e_i$, then $\sum_i e_i \in p_1 A p_1$, i.e. $\sup_{pB(H)p} e_i \in p_1 A p_1$. The last statement is a contradiction. Therefore $\{a_{\xi\eta}\} \notin A$. Hence $A \neq \sum_{\xi,\eta}^\oplus p_\xi A p_\eta$ (i.e. $\mathcal{A} := \{\{p_\xi a p_\eta\} : a \in A\} \neq \sum_{\xi,\eta}^\oplus p_\xi A p_\eta$). \triangleright

The next assertion follows by theorem 13 and it's proof.

Corollary 14. *Let A be a C^* -algebra on a Hilbert space H , $\{p_\xi\}$ be an infinite orthogonal set of projections in A with the least upper bound 1 in the algebra $B(H)$. Suppose that the order unit space $\sum_{\xi,\eta}^\oplus p_\xi A p_\eta$ is monotone complete and there exists a bounded monotone increasing net (a_α) of elements in $\sum_{\xi,\eta}^\oplus p_\xi A p_\eta$, the least upper bound $\sup_{\sum_{\xi,\eta}^\oplus p_\xi A p_\eta} a_\alpha$ in the algebra $\sum_{\xi,\eta}^\oplus p_\xi A p_\eta$ and the least upper bound $\sup_{\sum_{\xi,\eta}^\oplus p_\xi B(H) p_\eta} a_\alpha$ in the algebra $\sum_{\xi,\eta}^\oplus p_\xi B(H) p_\eta$ of which are different. Then the vector space $\sum_{\xi,\eta}^\oplus p_\xi A p_\eta$ is not closed concerning the multiplication of the algebra $\sum_{\xi,\eta}^\oplus p_\xi B(H) p_\eta$.*

2. APPLICATION

Let n be an infinite cardinal number, Ξ a set of indexes of cardinality n . Let $\{e_{ij}\}$ be a set of matrix units such that e_{ij} is a $n \times n$ -dimensional matrix, i.e. $e_{ij} = (a_{\alpha\beta})_{\alpha\beta \in \Xi}$, whose (i, j) -s component is 1, i.e. $a_{ij} = 1$, and the rest components are zero. Let X be a hyperstonean compact, $C(X)$ the commutative algebra of all complex-valued continuous functions on the compact X and

$$\mathcal{M} = \left\{ \sum_{ij \in \Xi} \lambda_{ij}(x) e_{ij} : (\forall ij \lambda_{ij}(x) \in C(X)) \right\}$$

$$(\exists K \in R)(\forall m \in N)(\forall \{e_{kl}\}_{kl=1}^m \subseteq \{e_{ij}\}) \left\| \sum_{kl=1 \dots m} \lambda_{kl}(x) e_{kl} \right\| \leq K \},$$

where $\left\| \sum_{kl=1 \dots m} \lambda_{kl}(x) e_{kl} \right\| \leq K$ means $(\forall x_o \in X) \left\| \sum_{kl=1 \dots m} \lambda_{kl}(x_o) e_{kl} \right\| \leq K$. The set \mathcal{M} is a vector space with pointwise algebraic operations. The map $\| \cdot \| : \mathcal{M} \rightarrow \mathbf{R}_+$ defined as

$$\|a\| = \sup_{\{e_{kl}\}_{kl=1}^n \subseteq \{e_{ij}\}} \left\| \sum_{kl=1}^n \lambda_{kl}(x) e_{kl} \right\|,$$

is a norm on the vector space \mathcal{M} , where $a \in \mathcal{M}$ and $a = \sum_{ij \in \Xi} \lambda_{ij}(x) e_{ij}$.

Theorem 15. \mathcal{M} is a von Neumann algebra of type I_n and $\mathcal{M} = C(X) \otimes M_n(\mathbf{C})$.

Proof. It is easy to see that the set \mathcal{M} is a vector space with the componentwise algebraic operations. It is known that the vector space $C(X, M_n(\mathbf{C}))$ of continuous matrix-valued maps on the compact X is a C^* -algebra. Let $A = C(X, M_n(\mathbf{C}))$

and e_i be a constant e_{ii} -valued map on X , i.e. e_i is a projection of the algebra A . Then $\{e_i\}$ is an orthogonal set of projections with $\sup_i e_i = 1$ in the algebra A . Then $\sum_{ij}^{\oplus} e_i A e_j = \mathcal{M}$. We have a C^* -algebra A can be embedded in $B(H)$ for some Hilbert space H . Then $\sum_{ij}^{\oplus} e_i A e_j$ can be embedded in $B(H)$. For any i $e_i A e_i = C(X)e_i$, i.e. the component $e_i A e_i$ is weakly closed in $B(H)$. Hence, by theorem 11 the image of vector space \mathcal{M} in $B(H)$ is a von Neumann algebra. Hence, \mathcal{M} is a von Neumann algebra. Note, that the set $\{e_i\}$ is a maximal orthogonal set of abelian projections with central support 1. Hence, \mathcal{M} is a von Neumann algebra of type I_n . Moreover the center $Z(\mathcal{M})$ of the algebra \mathcal{M} is isomorphic to $C(X)$ and $\mathcal{M} = C(X) \otimes M_n(\mathbf{C})$. \triangleright

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